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Planck's Energy Elements

What the 1900 derivation does, step by step

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Abstract. Planck's December 1900 derivation introduces a finite energy element so that Boltzmann's count can be performed, then uses Wien's displacement law to set that element proportional to frequency. The constant of proportionality is h. This paper writes the steps in order, separates them from the later phrase "act of desperation," and separates Planck's discreteness from the discreteness of a Fourier harmonic. The ultraviolet catastrophe is Ehrenfest's later name for the divergence of equipartition. It was not the sentence Planck was repairing on 19 October 1900. Nothing in the derivation times the emission from one resonator.

1. The quantity

A black body in equilibrium has a spectral energy density $u(\nu, T)$ that depends on frequency and temperature only. Kirchhoff had made the universality a theorem [1]. Wien's displacement law [2] and Wien's 1896 distribution [3] matched the short-wavelength data and failed in the infrared. The long-wavelength measurements of Rubens and Kurlbaum [12], and the earlier cavity work of Lummer and Pringsheim [17], are the experimental pressure on Planck. Rayleigh's 1900 equipartition argument [7], clarified by Jeans in 1905 [8], gives a density proportional to $\nu^2 T$. Integrated over frequency it diverges. Ehrenfest, in 1911 [9], is the one who fixed the phrase ultraviolet catastrophe on that divergence. Planck's October move is an interpolation against the data, under the constraint that the two laws of thermodynamics stay.

2. 19 October 1900

In "Über eine Verbesserung der Wien'schen Spectralgleichung," *Verhandlungen der Deutschen Physikalischen Gesellschaft* 2 (1900) 202 [4], Planck proposes a formula that bridges Wien's entropy relation and the long-wavelength side. Wien's law corresponds to a second derivative of resonator entropy S with respect to resonator energy U of the form $\partial^2 S/\partial U^2 = -1/(bU)$. He writes a more general expression whose consequences fit the new measurements better. He later called this October formula a happily guessed interpolation [10]. The guess is the shape. The count is not yet in the paper.

3. 14 December 1900

"Zur Theorie des Gesetzes der Energieverteilung im Normalspectrum," the same *Verhandlungen* 2 (1900) 237 [5], is the derivation. The Annalen text is *Annalen der Physik* 4 (1901) 553 [6]. The steps are these.

Electromagnetic theory already ties the radiation density to the mean energy U of a resonator of frequency ν, through the mode factor $8\pi\nu^2/c^3$. If U were kT, Rayleigh–Jeans would follow. The unsolved piece is U.

He adopts Boltzmann's relation $S = k \log W$ [13] as the connection between the entropy of N resonators and the number of realizations W.

To compute W he refuses to treat the total energy U_N as infinitely divisible. It is composed of a whole number P of equal finite parts. Each part is an energy element ε.

$$U_N = P\varepsilon.$$

The number of ways of distributing P indistinguishable elements among N distinguishable resonators is

$$W = (N + P - 1)!/(P!(N - 1)!).$$

Stirling's approximation produces an entropy. Matching it to the entropy implied by the October formula constrains ε.

Wien's displacement law, which he keeps, requires ε proportional to frequency. He writes

$$\varepsilon = h\nu,$$

with h a new universal constant. The mean resonator energy and the spectral density follow:

$$U = (h\nu)/(\exp(h\nu/kT) - 1),$$

$$u(\nu, T) = \left(8\pi h\nu^3/c^3\right)/(\exp(h\nu/kT) - 1).$$

The 1901 paper gives $h = 6.55 \times 10^{-27}$ erg·s and $k = 1.346 \times 10^{-16}$ erg/deg, from the spectra then in hand. Later digits differ. The logic does not.

4. What sort of discreteness

The element $h\nu$ chops the energy at one frequency. It does not chop the frequency axis into harmonics of a fundamental. The resonators are still indexed by a continuous ν, with a smooth density of modes. A high-frequency resonator is dark at modest T because $h\nu$ is large next to kT, so the count rarely assigns it even one element. A cutoff produced by summing $1/n^2$ over the overtones of a single tone is a different mathematical fact and does not appear in this derivation.

Kuhn's historical warning belongs here [15]. The 1900 and 1901 papers [5, 6] say the total energy, for the count, is composed of elements. They do not contain a clear additional sentence that one oscillator is physically forbidden from every intermediate energy. Einstein's 1906 paper [14] is the one that treats the derivation as requiring resonator energies $0, h\nu, 2h\nu, \ldots$. Planck discretized the count. He did not yet say what Einstein said he had discretized.

5. What he later said

The phrase "act of desperation" is from Planck's letter to Robert Williams Wood, 7 October 1931 [10], not from the 1900 lecture. He writes that a theoretical interpretation had to be found at any cost, that he would sacrifice his other convictions but not the two laws of thermodynamics, and that assuming energy is forced to remain together in quanta was a formal assumption, adopted to get a positive result. That is a recollection. It describes his reluctance. It does not delete the December argument [5].

Around 1911, in the so-called second theory [16], he tried to confine the discontinuity to emission and leave absorption continuous, so the vacuum could stay Maxwellian. He did not recover a classical equipartition derivation of the same law. In 1911 he also said the hypothesis of quanta would never vanish. Permanence of h is not a theory of the instant at which one resonator emits, and it is not agreement with Einstein's light quantum [14].

6. What this does not do

The derivation assigns a mean energy. It does not assign the moment at which a particular resonator loses an element. A probability, or a rate, can be laid over the mean afterwards. Planck does not, in these papers, possess the individual law Einstein will still be demanding in 1949. The constant h is the size of the element that made the count match the spectrum. It is the start of the discretized method in that precise sense. It is not a heuristic Planck noticed and set aside. The October formula was the guess. The December element was the cost of turning the guess into a count. He hoped, later, that the cost was only formal. He never paid for the spectrum in a purely continuous currency.

7. The second derivative, written out

The October interpolation is not a guess about a sine series. It is a bridge between two thermodynamic limits. With $1/T = \partial S/\partial U$,

$$\text{Wien: } \partial^2 S/\partial U^2 = -k/(\varepsilon U),$$

$$\text{Rayleigh–Jeans: } \partial^2 S/\partial U^2 = -k/U^2,$$

$$\text{Planck: } \partial^2 S/\partial U^2 = -k/(U(U+\varepsilon)).$$

One integration gives $U = \varepsilon/\left(e^{\varepsilon/kT} - 1\right)$. Displacement law [2] sets $\varepsilon = h\nu$. The mode factor $8\pi\nu^2/c^3$ then produces the density. The experimenters who fixed the two ends of the bridge are Lummer and Pringsheim [17] and Rubens and Kurlbaum [12]. Rayleigh [7] and Jeans [8] fixed the long-wave formula he had to recover. Ehrenfest's name for the divergent integral comes in 1911 [9].

8. What this expansion does not add

A fuller account of the same papers does not turn the element into a harmonic index, and it does not produce the emission instant. Poincaré, after Solvay 1911, argued that some discontinuity in the dynamics is required by the law itself [11]. Planck's second theory [16] tried to put that discontinuity into emission only. It did not restore equipartition. The History tab carries the later camps. They are not hidden inside the 1900 count.

9. References

Numbers in the text refer to this list.

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3. W. Wien, "Über die Energievertheilung im Emissionsspectrum eines schwarzen Körpers," *Annalen der Physik* 58, 662–669 (1896).
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8. J. H. Jeans, "On the partition of energy between matter and aether," *Philosophical Magazine* 10, 91–98 (1905).
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14. A. Einstein, "Zur Theorie der Lichterzeugung und Lichtabsorption," *Annalen der Physik* 20, 199–206 (1906).
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16. M. Planck, "Eine neue Strahlungshypothese," *Verhandlungen der Deutschen Physikalischen Gesellschaft* 13, 138–148 (1911).

17. O. Lummer and E. Pringsheim, “Die Vertheilung der Energie im Spectrum des schwarzen Körpers,” *Verhandlungen der Deutschen Physikalischen Gesellschaft* 1, 23–41 (1899).

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